2015 Che2410 – Homework Assignment #1
Due on Sept. 22nd at 4:30pm

1. (8 pts) Classify the following differential equations. State whether they are linear/nonlinear, ordinary/partial, homogeneous/inhomogeneous and state their order. If applicable, state whether the equation is elliptic, hyperbolic, or parabolic, and under what conditions (if any).

a) $f''' + f' = 0$
Linear, ordinary, 3rd order, homogeneous

b) $\frac{dy}{dx} + x^2 \frac{d^2y}{dx^2} = \frac{1}{x}$
Linear, ordinary, 2nd order, inhomogeneous

c) $f_t f_x = 1$
Nonlinear, partial, 1st order, neither homogeneity or elliptic/hyperbolic/parabolic are applicable (because it is not linear)

d) $y(x) + y'(x) + xy''(x) = 0$
Linear, ordinary, 2nd order, homogeneous

e) $(x^2 + 4)dy = (2x - 8xy)dx$
Linear, ordinary, 1st order, inhomogeneous

f) $2f_{xx} + 4f_{xy} - f_{yy} + f_x = 0$
Linear, partial, 2nd order, homogeneous, hyperbolic

g) $x\frac{\partial^2 y}{\partial x^2} + \frac{\partial^2 y}{\partial t^2} + 2\frac{\partial^2 y}{\partial x \partial t} = 1$
Linear, partial, 2nd order, inhomogeneous, parabolic if x = 1, elliptic if x < 1, hyperbolic if x > 1

h) $\nabla^2 \phi = 0$
Linear, partial, 2nd order, homogenous (ordinary is also acceptable, since eqn doesn't specify # of dimensions)

2. (4 pts) Solve the following 1st order differential equations:

a) $x^2 \frac{df}{dx} + 4f = 2$

You can use a variety of methods to get the answer, including the integrating factor method:

$$f = \frac{1}{2} + Ce^{\frac{4}{x}}$$

b) $(x^2 + 4)\frac{dy}{dx} + 8xy = 2x$

You can use a variety of methods to get the answer, including the integrating factor method:

$$f = \frac{1}{4} + \frac{C}{(4 + x^2)^4}$$

3. (4 pts) Derive the formula for the 6th order central difference operator. Determine the leading order term in the truncation error.

Expand $f(x+\Delta x)$ and $f(x-\Delta x)$ in Taylor series, subtract one from the other and solve for f_x :

$$(A)\ f_x = \frac{f(x+\Delta x) - f(x-\Delta x)}{2\ \Delta x} - \frac{1}{6} f_{xxx} \Delta x^2 - \frac{1}{120} f_{xxxxx} \Delta x^4 - \frac{1}{5040} f_{xxxxxxx} \Delta x^6 - O(\Delta x^8)$$

Note that we have to expand at least this far out in order to find the leading order term in the truncation error.

The above is just as we derived in class. Now substitute into the above equation $2\Delta x$ and $3\Delta x$:

$$(B)\ f_x = \frac{f(x+2\Delta x) - f(x-2\Delta x)}{4\ \Delta x} - \frac{4}{6} f_{xxx} \Delta x^2 - \frac{16}{120} f_{xxxxx} \Delta x^4 - \frac{64}{5040} f_{xxxxxxx} \Delta x^6 - O(\Delta x^8)$$

$$(C)\ f_x = \frac{f(x+3\Delta x) - f(x-3\Delta x)}{6\ \Delta x} - \frac{9}{6} f_{xxx} \Delta x^2 - \frac{81}{120} f_{xxxxx} \Delta x^4 - \frac{729}{5040} f_{xxxxxxx} \Delta x^6 - O(\Delta x^8)$$

For each equation individually, if we dropped the terms with unknown derivatives (e.g., f_{xxx} and higher derivatives) our resulting approximation would have Δx^2 in the leading terms of the truncation error. The goal is to end up with an approximation for f_x that has Δx^6 as the lowest exponent of Δx in the dropped (truncated) terms.

We can do this by treating A, B, and C, as a system of 3 equations with 3 unknowns, where the unknowns are the factors that we need to multiply A, B, and C, by so that the Δx^2 and Δx^4 terms cancel out.

The result is :

$$f_x = 3\frac{f(x+\Delta x) - f(x-\Delta x)}{4\ \Delta x} - 3\frac{f(x+2\Delta x) - f(x-2\Delta x)}{20\ \Delta x} + \frac{f(x+3\Delta x) - f(x-3\Delta x)}{60\ \Delta x} - \frac{1}{140} f_{xxxxxxx} \Delta x^6 - O(\Delta x^8)$$

Thus the leading order term in the truncation error is $-\frac{1}{140} f_{xxxxxxx} \Delta x^6$

4. (4 pts) Derive the 4th order approximation to the second derivative. Determine the leading order term in the truncation error.

Expand $f(x+\Delta x)$ and $f(x-\Delta x)$ in Taylor series:

$$f(x+\Delta x) = f(x) + f_x\Delta x + \frac{1}{2}f_{xx}\Delta x^2 + \frac{1}{6}f_{xxx}\Delta x^3 + \frac{1}{24}f_{xxxx}\Delta x^4 + \frac{1}{120}f_{xxxxx}\Delta x^5 + O(\Delta x^6)$$

$$f(x-\Delta x) = f(x) - f_x\Delta x + \frac{1}{2}f_{xx}\Delta x^2 - \frac{1}{6}f_{xxx}\Delta x^3 + \frac{1}{24}f_{xxxx}\Delta x^4 - \frac{1}{120}f_{xxxxx}\Delta x^5 + O(\Delta x^6)$$

Notice that by *adding* these two equations, the first derivatives cancel. So if we add them and solve for f_{xx} we get:

$$f_{xx} = \frac{f(x+\Delta x) - 2f(x) + f(x-\Delta x)}{\Delta x^2} - \frac{1}{12}\mathrm{f}_{\mathrm{xxxx}}\Delta x^2 - \frac{1}{360}f_{xxxxxx}\Delta x^4 + O(\Delta x^6)$$

We can now substitute $2\Delta x$ into the above to get another equation:

$$f_{xx} = \frac{f(x+2\Delta x) - 2f(x) + f(x-2\Delta x)}{4\Delta x^2} - \frac{4}{12}\mathrm{f}_{\mathrm{xxxx}}\Delta x^2 - \frac{16}{360}f_{xxxxxx}\Delta x^4 + O(\Delta x^6)$$

By inspection one can see that subtracting 4x the first equation from the second will eliminate the Δx^2 term, resulting in the answer:

$$f_{xx} = \frac{4}{3}\frac{f(x+\Delta x) - 2f(x) + f(x-\Delta x)}{\Delta x^2} - \frac{1}{3}\frac{f(x+2\Delta x) - 2f(x) + f(x-2\Delta x)}{4\Delta x^2} + \frac{1}{90}f_{xxxxxx}\Delta x^4 + O(\Delta x^6)$$

Hence the leading order term in the truncation error is $\frac{1}{90}f_{xxxxxx}\Delta x^4$.

5. (4 pts) Derive the formulas for the 2nd and 3rd order forward differencing.

Expand $f(x+\Delta x)$ in Taylor series and solve for f_x:

$$f_x = \frac{f(x+\Delta x) - f(x)}{\Delta x} - \frac{1}{2}\mathrm{f_{xx}}\Delta x - \frac{1}{6}f_{xxx}\Delta x^2 - \frac{1}{24}f_{xxxx}\Delta x^3 + O(\Delta x^4)$$

Now substitute $2\Delta x$ into the above equation:

$$f_x = \frac{f(x+2\Delta x) - f(x)}{2\Delta x} - \mathrm{f_{xx}}\Delta x - \frac{4}{6}f_{xxx}\Delta x^2 - \frac{8}{24}f_{xxxx}\Delta x^3 + O(\Delta x^4)$$

Subtracting 2x the first equation from the second eliminates the Δx term, yielding:

$$f_x = -\frac{3f(x)}{2\Delta x} + 2\frac{f(x+\Delta x)}{\Delta x} - \frac{f(x+2\Delta x)}{2\Delta x} + \frac{1}{3}\mathrm{f_{xxx}}\Delta x^2 + O(\Delta x^3)$$

Where the leading term is $\frac{1}{3}\mathrm{f_{xxx}}\Delta x^2$

Similarly, by substituting $3\Delta x$ into the top equation, and then solving a system such that the Δx and Δx^2 terms disappear, we get:

$$f_x = -\frac{11f(x)}{6\Delta x} + 3\frac{f(x+\Delta x)}{\Delta x} - 3\frac{f(x+2\Delta x)}{2\Delta x} + \frac{f(x+3\Delta x)}{3\Delta x} + \frac{1}{4}\mathrm{f_{xxxx}}\Delta x^3 + O(\Delta x^4)$$

Where the leading term is $\frac{1}{4}\mathrm{f_{xxxx}}\Delta x^3$

6. (6 pts) Consider the equation:

$$f_t = f_x$$

Show that the MacCormack scheme is algebraically equivalent to the Lax-Wendroff scheme.

Lax-Wendroff:

$$f_j^{n+1} = f_j^n + \frac{\lambda}{2}\left(f_{j+1}^n - f_{j-1}^n\right) + \frac{\lambda^2}{2}\left(f_{j+1}^n + f_{j-1}^n - 2f_j^n\right)$$

MacCormack (FB variant):

Predictor: $\hat{f}_j = f_j^n + \lambda\left(f_{j+1}^n - f_j^n\right)$

Corrector: $f_j^{n+1} = \frac{1}{2}\left(\hat{f}_j + f_j^n + \lambda\left(\hat{f}_j - \hat{f}_{j-1}\right)\right)$

Substitute predictor into corrector:

$$f_j^{n+1} = \frac{1}{2}\left(f_j^n + \lambda\left(f_{j+1}^n - f_j^n\right) + f_j^n + \lambda\left(f_j^n + \lambda\left(f_{j+1}^n - f_j^n\right) - \left(f_{j-1}^n + \lambda\left(f_j^n - f_{j-1}^n\right)\right)\right)\right)$$

$$= f_j^n + \frac{\lambda}{2}\left(f_{j+1}^n - f_{j-1}^n\right) + \frac{\lambda^2}{2}\left(f_{j+1}^n + f_{j-1}^n - 2f_j^n\right) \leftarrow \text{Lax-Wendroff}$$