2015 Che2410 – Homework Assignment #4
Due on Nov. 13th at midnight

1. Solve the following differential equations, please show your work.

a) $y'' + 6y' + 5y = 0$

$$m^2 + 6m + 5 = 0$$
$$(m+1)(m+5) = 0$$
$$y(x) = c_1 e^{-x} + c_2 e^{-5x}$$

b) $y'' + 6y' + 9y = 0$

$$m^2 + 6m + 9 = 0$$
$$(m+3)^2 = 0$$
$$y(x) = c_1 e^{-3x} + c_2 x e^{-3x}$$

c) $y'' + 6y' + 13y = 0$

$$m^2 + 6m + 13 = 0$$
$$m = \frac{-6 \pm \sqrt{6^2 - 4*13)}}{2}$$
$$= -3 \pm 2i$$
$$y(x) = c_1 e^{(-3+2i)x} + c_2 e^{(-3-2i)x}$$

Or

$$y(x) = e^{-3x}(c_1 \cos 2x + c_2 \sin 2x)$$

d) $y'' - 3y' + 2y = xe^x + xe^{-x}$

Homogeneous equation:

$$y_H'' - 3y_H' + 2y_H = 0$$
$$y_H(x) = c_1 e^x + c_2 e^{2x}$$

Use variation of parameters to get particular solution:

$$W = \begin{vmatrix} e^x & e^{2x} \\ e^x & 2e^{2x} \end{vmatrix} = e^{3x}$$
$$y_P(x) = -y_1 \int \frac{f y_2}{W a_0} dx + y_2 \int \frac{f y_1}{W a_0} dx$$
$$= e^x \int \frac{(xe^x + xe^{-x})e^{2x}}{e^{3x}} dx$$
$$+ e^{2x} \int \frac{(xe^x + xe^{-x})e^x}{e^{3x}} dx$$
$$y_P(x) = -xe^x - \frac{1}{2}x^2 e^x + \frac{5}{36}e^{-x} + \frac{1}{6}xe^{-x}$$
$$y(x) = y_H(x) + y_P(x)$$

e) $y'' + 4y' + 4y = e^{-2x}/x^2$

Homogeneous eqation:

$$y_H'' + 4y_H' + 4y_H = 0$$
$$(m+2)^2 = 0$$
$$y_H(x) = c_1 e^{-2x} + c_2 x e^{-2x}$$

Use variation of parameters to get particular solution:

$$W = \begin{vmatrix} e^{-2x} & xe^{-2x} \\ -2e^{-2x} & (1-2x)e^{-2x} \end{vmatrix} = e^{-4x}$$
$$y_P = -e^{-2x} \ln x - e^{-2x}$$
$$y(x) = y_H(x) + y_P(x)$$

f) $(x-1)y'' - xy' + y = 0$

One can guess that $y_1 = e^x$ (always a good idea to guess the really simple solutions: $y = 0, 1, x, e^x$).

After that, use reduction of order:

$$y_2 = y_1 \int \frac{dx}{p\, y_1^2} dx = e^x \int \frac{x-1}{e^{2x} e^{1-x}} dx$$
$$= -x/e$$
$$y(x) = c_1 e^x + c_2 x$$

2. Solve (if possible) the following problems. If necessary, indicate a solvability condition. Comment on the uniqueness of the solution.

(Note that, if you can show using the Fredhold Alternative Theorem that there is no solution, you do not need to do any more work on the solution. Similarly, you may want to use a theorem proves that a solution to the initial-value problem is unique).

a) $y''' - 2y'' + y' = 1 + xe^x \quad (x > 0); \quad y(0) = 0, y'(0) = 0, y''(0) = 1$

Let $z = y'$ yielding:

$$z'' - 2z' + z = 1 + xe^x$$
$$\therefore z_H = c_1 e^x + c_2 x e^x$$

Use variation of parameters to get z_P

$$W = \begin{vmatrix} e^x & xe^x \\ e^x & (1+x)e^x \end{vmatrix} = e^{2x}$$

$$z_P(x) = -z_1 \int \frac{f z_2}{W a_0} dx + z_2 \int \frac{f z_1}{W a_0} dx$$
$$= -e^x \int \frac{(1 + xe^x) x e^x}{e^{2x}} dx + xe^x \int \frac{(1 + xe^x) e^x}{e^{2x}} dx$$
$$= 1 + \frac{1}{6} x^3 e^x$$

$$\therefore z = c_1 e^x + c_2 x e^x + 1 + \frac{1}{6} x^3 e^x$$

Now integrate z to get y:

$$y = c_1 e^x + c_2 x e^x + c_3 + x - \frac{1}{2} x^2 e^x + \frac{1}{6} x^3 e^x$$

Finally apply initial value conditions:

$$y(0) = c_1 + c_3 = 0$$
$$y('0) = c_1 + c_2 + 1 = 0$$
$$y''(0) = -1 + c_1 + 2c_2 = 1$$

$$\therefore c_1 = -4, c_2 = 3, c_3 = 4$$

$$\boxed{y(x) = 4 - 4e^x + x + 3xe^x - \frac{1}{2} x^2 e^x + \frac{1}{6} x^3 e^x}$$

b) $e^x y''' + \sin x\, y'' + \cos x\, y' + x^6 y = 0 \quad (x > 0); \quad y(0) = 0, y'(0) = 0, y''(0) = 0$

For this problem, one can guess that $y(x) = 0$ is a solution. Since it is an initial value problem, $y(x) = 0$ is the unique, and only, solution.

c) $xy'' - (x+1)y' + y = x^2 e^{2x} \quad (x > 1); \quad y(1) = 0, y'(1) = e$

Hint: $y(x) = x + 1$ is a solution to the homogeneous equation

Solve homogeneous case first:

$y_1 = x + 1$, (given), use reduction of order for y_2:

$$y_2 = y_1 \int \frac{dx}{p\, y_1^2} dx = (x+1) \int \frac{x\, e^x\, dx}{(x+1)^2} dx$$

Use integration by parts and problem hint if necessary:

$$= \frac{e^x (x+1)}{x+1} = e^x$$

$y_H = c_1(x+1) + c_2 e^x$

Use variation of parameters to get the particular solution:

$$W = \begin{vmatrix} (x+1) & e^x \\ 1 & e^x \end{vmatrix} = xe^x$$

$$\begin{aligned} y_P(x) &= -y_1 \int \frac{f y_2}{W a_0} dx + y_2 \int \frac{f y_1}{W a_0} dx \\ &= -(x+1) \int \frac{x^2 e^{2x} e^x}{x e^x x} dx + e^x \int \frac{x^2 e^{2x}(x+1)}{x e^x x} dx \\ &= -\frac{1}{2} e^{2x} + \frac{1}{2} x\, e^{2x} \end{aligned}$$

$$y(x) = c_1(x+1) + c_2 e^x + (x-1)\frac{1}{2} x e^{2x}$$

Applying the initial value condutions:

$y(1) = 2c_1 + ec_2 = 0$

$$y'(1) = c_1 + ec_2 + \frac{1}{2} e^2 = e$$

$c_1 = \frac{1}{2} e^2 - e, c_2 = 2 - e$

$$\therefore \boxed{y(x) = \left(\frac{1}{2} e^2 - e\right)(x+1) + (2-e)e^x + (x-1)\frac{1}{2} x e^{2x}}$$