2016 Che2410 – Homework Assignment #1
Due on Sept. 27th at 6:30pm

1. (10 pts) Classify the following differential equations. If applicable, state whether the equation is linear/nonlinear, ordinary/partial, homogeneous/inhomogeneous, the order, and whether the equation is elliptic, hyperbolic, or parabolic, and under what conditions (if any).

a) $f''' = f$
Linear, ordinary, homogeneous, 3rd order

b) $y + \frac{dy}{dx} + x^2 \frac{d^2y}{dx^2} = x$
Linear, ordinary, inhomogeneous, 2nd order

c) $f_t f_{xx} = 1$
Non-linear, partial, homo/inhomo not applicable, 2nd order in x – 1st order in t

d) $\sqrt{y(x)} + y'(x) + xy''(x) = 5x$
Non-linear, ordinary, homo/inhomo not applicable, 2nd order

e) $(y+1)dy = (4 - 4xy)dx$
Non-linear, ordinary, 1st order

f) $f_{xx} + f_{xy} = f_{yy}$
Linear, partial, homogeneous, hyperbolic

g) $\frac{\partial^2 y}{\partial x^2} + x\frac{\partial^2 y}{\partial t^2} + t\frac{\partial^2 y}{\partial x \partial t} = y$
Linear, partial, homogeneous, 2nd order in x and t, elliptic if $t^2 < 4x$, hyperbolic if $t^2 > 4x$, and parabolic if $t^2 = 4x$

h) $\nabla^2 \phi = f$
Linear, partial or ordinary (dimensionality was not well specified), inhomogeneous, 2nd order

i) $\sin(f) = f'$
Nonlinear, ordinary, homo/inhomo not applicable, 1st order

j) $\tan(xy)\frac{dz}{dx} + \frac{dz}{dy} = 0$
Linear, partial, homogeneous, parabolic

2. (10 pts) Solve the following 1st order differential equations:

Use integrating factor method for both problems:

a) $x^3 \frac{df}{dx} + 4f = 2f - 5$

$$f = Ce^{\frac{1}{x^2}} - 5/2$$

b) $8xf + (x^2 + 4)f' = x$

$$f = \frac{C}{(4+x^2)^4} + \frac{1}{8}$$

3. (10 pts) Derive a formula that is 4th order in error for:

(a) a forward difference operator
(b) a central difference operator

In each case, determine the leading order term in the truncation error.

(a) Expand $f(x+\Delta x)$ in Taylor series and solve for f_x

$$(A)\ f_x = \frac{f(x+\Delta x)-f(x)}{\Delta x} - \frac{1}{2}\mathrm{f_{xx}}\Delta x - \frac{1}{6}f_{xxx}\Delta x^2 - \frac{1}{24}f_{xxxx}\Delta x^3 - \frac{1}{120}f_{xxxxx}\Delta x^4 + +O(\Delta x^5)$$

There are several ways to create a 4th order forward difference operator (in other words, there is not a unique solution). We will go with the arguably simplest solution by considering $f(x+2\Delta x)$, $f(x+3\Delta x)$, and $f(x+4\Delta x)$. In each case, start with equation (A) above and substitute Δx with $N\Delta x$ where N is (2,3,4, etc.) to get the following equations:

$$(B)\ f_x = \frac{f(x+2\Delta x)-f(x)}{2\Delta x} - \mathrm{f_{xx}}\Delta x - \frac{4}{6}f_{xxx}\Delta x^2 - \frac{8}{24}f_{xxxx}\Delta x^3 - \frac{16}{120}f_{xxxxx}\Delta x^4 + +O(\Delta x^5)$$

$$(C)\ f_x = \frac{f(x+3\Delta x)-f(x)}{3\Delta x} - \frac{3}{2}\mathrm{f_{xx}}\Delta x - \frac{9}{6}f_{xxx}\Delta x^2 - \frac{27}{24}f_{xxxx}\Delta x^3 - \frac{81}{120}f_{xxxxx}\Delta x^4 + +O(\Delta x^5)$$

$$(D)\ f_x = \frac{f(x+4\Delta x)-f(x)}{4\Delta x} - \frac{4}{2}\mathrm{f_{xx}}\Delta x - \frac{16}{6}f_{xxx}\Delta x^2 - \frac{64}{24}f_{xxxx}\Delta x^3 - \frac{256}{120}f_{xxxxx}\Delta x^4 + +O(\Delta x^5)$$

For each equation individually, if we dropped the terms with unknown derivatives (e.g., f_{xx} and higher derivatives) our resulting approximation would have Δx in the leading terms of the truncation error. The goal is to end up with an approximation for f_x that has Δx^4 as the lowest exponent of Δx in the dropped (truncated) terms.

We can do this by treating A, B, C, and D as a system of 4 equations with 4 unknowns, where the unknowns are the factors that we need to multiply A, B, C, and D by so that the required terms cancel out.

The result is:

$$(A)\ f_x \approx 4\frac{f(x+\Delta x)-f(x)}{\Delta x} - 6\frac{f(x+2\Delta x)-f(x)}{2\Delta x} + 4\frac{f(x+3\Delta x)-f(x)}{3\Delta x} - \frac{f(x+4\Delta x)-f(x)}{4\Delta x}$$

where the leading order error term being $-\frac{1}{5}f_{xxxxx}\Delta x^4$

(b) Expand $f(x+\Delta x)$ and $f(x-\Delta x)$ in Taylor series, subtract one from the other and solve for f_x :

$$(A)\ f_x = \frac{f(x+\Delta x)-f(x-\Delta x)}{2\ \Delta x} - \frac{1}{6} f_{xxx} \Delta x^2 - \frac{1}{120} f_{xxxxx} \Delta x^4 - O(\Delta x^6)$$

Note that we have to expand at least this far out in order to find the leading order term in the truncation error.

The above is just as we derived in class. Now substitute into the above equation $2\Delta x$:

$$(B)\ f_x = \frac{f(x+2\Delta x)-f(x-2\Delta x)}{4\ \Delta x} - \frac{4}{6} f_{xxx} \Delta x^2 - \frac{16}{120} f_{xxxxx} \Delta x^4 - O(\Delta x^6)$$

For each equation individually, if we dropped the terms with unknown derivatives (e.g., f_{xxx} and higher derivatives) our resulting approximation would have Δx^2 in the leading terms of the truncation error. The goal is to end up with an approximation for f_x that has Δx^4 as the lowest exponent of Δx in the dropped (truncated) terms.

We can do this by treating A and B as a system of 2 equations with 2 unknowns, where the unknowns are the factors that we need to multiply A and B by so that the Δx^2 terms cancel out.

The solution is:

$$f_x \approx \frac{4}{3} \frac{f(x+\Delta x)-f(x-\Delta x)}{2\ \Delta x} - \frac{1}{3} \frac{f(x+2\Delta x)-f(x-2\Delta x)}{4\ \Delta x}$$

where the leading order error term is $-\frac{1}{30} f_{xxxxx} \Delta x^4$.

4. (10 pts) Derive a 3rd order approximation to the second derivative using a backward difference operator. Determine the leading order term in the truncation error.

Expand a Taylor series of $f(x+\Delta x)$ around x

$$(A^*)\ f(x-\Delta x) = f(x) - f_x \Delta x + \frac{1}{2} f_{xx} \Delta x^2 - \frac{1}{6} f_{xxx} \Delta x^3 + \frac{1}{24} f_{xxxx} \Delta x^4 - \frac{1}{120} f_{xxxxx} \Delta x^5 + +O(\Delta x^6)$$

Since we are interested in isolating f_{xx} divide all terms by Δx^2

$$(A) \frac{f(x-\Delta x)}{\Delta x^2} = \frac{f(x)}{\Delta x^2} - \frac{f_x}{\Delta x} + \frac{1}{2} f_{xx} - \frac{1}{6} f_{xxx} \Delta x + \frac{1}{24} f_{xxxx} \Delta x^2 - \frac{1}{120} f_{xxxxx} \Delta x^3 + +O(\Delta x^4)$$

So now we need to eliminate the f_x term (since we don't know what it is) and all of the terms whose error scales with Δx or Δx^2. We need more equations!

$$(B) \frac{f(x-2\Delta x)}{4\Delta x^2} = \frac{f(x)}{4\Delta x^2} - \frac{f_x}{2\Delta x} + \frac{1}{2} f_{xx} - \frac{2}{6} f_{xxx} \Delta x + \frac{4}{24} f_{xxxx} \Delta x^2 - \frac{8}{120} f_{xxxxx} \Delta x^3 + +O(\Delta x^4)$$

$$(C) \frac{f(x-3\Delta x)}{9\Delta x^2} = \frac{f(x)}{9\Delta x^2} - \frac{f_x}{3\Delta x} + \frac{1}{2} f_{xx} - \frac{3}{6} f_{xxx} \Delta x + \frac{9}{24} f_{xxxx} \Delta x^2 - \frac{27}{120} f_{xxxxx} \Delta x^3 + +O(\Delta x^4)$$

$$(D) \frac{f(x-4\Delta x)}{16\Delta x^2} = \frac{f(x)}{16\Delta x^2} - \frac{f_x}{4\Delta x} + \frac{1}{2} f_{xx} - \frac{4}{6} f_{xxx} \Delta x + \frac{16}{24} f_{xxxx} \Delta x^2 - \frac{64}{120} f_{xxxxx} \Delta x^3 + +O(\Delta x^4)$$

As before, by solving this system of 4 equations a 4 unknowns you can eliminate all of the terms you don't want and isolate the f_{xx} term. The coefficient of the leading order term is $\frac{5}{6} \Delta x^3$.

5. (10 pts) Consider the equation:

$$u_t = u_{xx} + u_{xxxx}$$

Is this a well-posed problem? Can it be solved using numerical methods?
(Some background: https://en.wikipedia.org/wiki/Well-posed_problem)

Solution:
Assume $u(x,t) = e^{\lambda t} e^{ikx}$, then we get

$$\lambda e^{\lambda t} e^{ikx} = -k^2 e^{\lambda t} e^{ikx} + k^4 e^{\lambda t} e^{ikx}$$

So $\lambda = -k^2 + k^4$, and $u(x,t) = e^{(-k^2+k^4)t} e^{ikx}$ which grows for all wave numbers k. Therefore, this problem is ill-posed. Note: it would be well posed if the sign in front of u_{xxxx} were negative.